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Sound produced when a vortex pair is swallowed by a jet

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Abstract. An analysis is made of the sound generated during the high-Reynolds-number convection of a vortex pair in a jet of water exhausting from a large vessel through a slit aperture. The equations of motion are linearized about the classical free-streamline solution describing steady flow through the aperture. It is assumed that the vortex pair is swept through the aperture into the jet by the steady mean flow, with no account taken of the nonlinear influence on the motion of 'images' in the boundaries. Additional vorticity is shed from the edges of the aperture in order that the flow should remain smooth and continuous (the Kutta condition). This vorticity is convected away within a sheet of 'bound' vorticity on the mean free streamlines of the jet. A strong peak in the bound vorticity is established when the vortex pair enters the aperture. Both the incident and the shed vorticity generate sound, but their respective contributions to the acoustic pressure are of *opposite* phase. The dominant radiation in the water above the aperture is produced as the vortex enters the jet, and has the form of a pressure pulse of width $\sim h/M$ and monopole strength $\sim \rho_o v^2/\sqrt{M}$, where h is the width of the aperture, ρ_o the density of the water, v a typical flow velocity, and M is the jet Mach number.

Key words: aerodynamic sound, cavitated flow, compact Green's function, unsteady free-streamlines, vortex sound

1. Introduction

Consider an infinite expanse of water in the region $x_2 > 0$ of the coordinate system (x_1, x_2, x_3). The water is at rest at infinity and the lower boundary $(x_2 = 0)$ is formed by a thin, rigid wall except for a two-dimensional aperture of width $2h$ occupying the interval $-h < x_1 < h$. The water drains steadily through the aperture into a nominally infinite vacuous or air-filled *cavity* and forms a jet of asymptotic width $2\sigma h$ and speed U, as indicated in Figure 1. When the fluid is regarded as incompressible, and at sufficiently high Reynolds number and Froude number U^2/gh ($g =$ acceleration due to gravity), the details of this flow (including the value $\sigma \approx 0{\cdot}61$ of the *contraction* coefficient) can be determined by classical *free streamline* theory [1, Section 73], [2, Section 6.13], [3, Section II.5], [4, Section II.C]. In this case the outflow is irrotational, except for vorticity shed steadily from the edges of the aperture and which is subsequently convected along the free-streamline boundaries of the jet.

A distant observer in the upper region will view the aperture as a steady *sink* of strength $2\sigma hU$ per unit length (in the x_3-direction). The sink strength will remain constant provided the mean pressure in the lower region does not vary with time, and provided the liquid flowing into the sink remains homogeneous. Many practical flows of this kind frequently contain inhomogeneities in the form of isolated distributions of vorticity. The existence of this vorticity indicates the presence of local 'pockets' of kinetic energy that characterize, for example, residual turbulence motions caused by 'stirring' or some other form of agitation or instability of the fluid entering the upper region. The rapid contraction of an eddy as it accelerates into the drain must be expected to cause a temporary fluctuation in the mass flux into the drain. This fluctuation is communicated to the distant observer in the form of sound waves

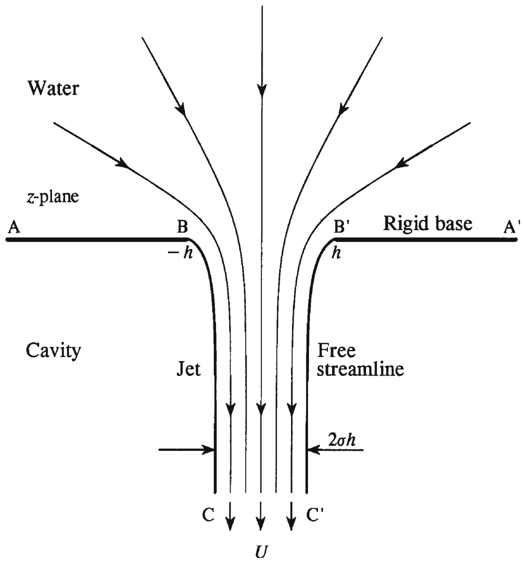

Figure 1. Nominally steady draining of water from an aperture.

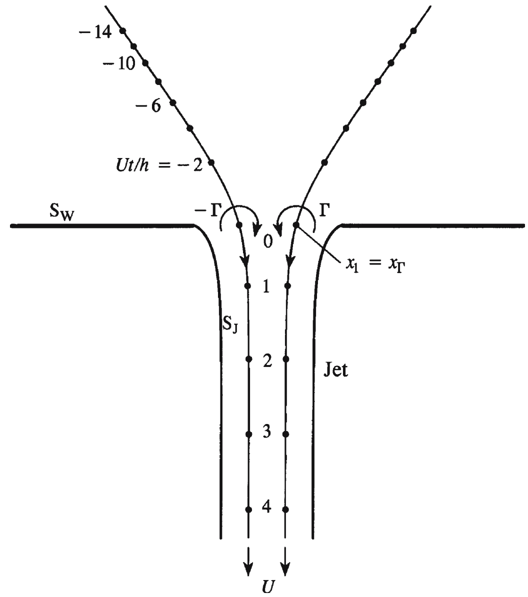

Figure 2. Numerical solution of Equations (8) defining the motion of the vortices $\pm\Gamma$ into the jet when $x_\Gamma = 0{\cdot}375h$. The vortices pass through the points • at the indicated nondimensional times Ut/h, measured from the instant at which the vortex pair crosses the plane of the aperture.

that can be attributed to a *monopole* acoustic source whose strength is just equal to the time rate of change of the mass flux [1, Section 73], [5], [6, Section 3.2]. Free-field turbulence forms a source of *quadrupole* type and produces far-field acoustic pressures of amplitude $\sim\rho_o v^2 M^2$, where ρ_o is the mean water density, v is a turbulence velocity, and $M = v/c_o \ll 1$ is the turbulence Mach number ($c_o =$ mean sound speed in the water) [5], [6, Section 2.1]. On the other hand, turbulence in accelerated motion near a rigid surface usually generates more intense *dipole* acoustic amplitudes $\sim\rho_o v^2 M$. But the unsteady mass flux produced when turbulence is swallowed by flow from an aperture in a boundary should generate yet more intense sound of monopole amplitude $\sim\rho_o v^2$.

These order-of-magnitude estimates are strictly applicable to three-dimensional flows. The corresponding acoustic-pressure estimates when the two-dimensional flow of Figure 1 is disturbed by two-dimensional turbulence would be $\rho_o v^2 M^{\frac{3}{2}}$, $\rho_o v^2 M^{\frac{1}{2}}$ and $\rho_o v^2 M^{-\frac{1}{2}}$, respectively, for quadrupole, dipole and monopole turbulence sources [5], [6, Chapter 2]. In either case the monopole contribution from the drain can constitute a very loud source of sound.

In this paper the simplest possible model problem is examined analytically to determine the properties of the monopole sound generated when an ‘eddy’ in the form of a vortex pair exhausts symmetrically through the aperture (Figure 2). The pressure field above the aperture will change as the vortex pair approaches, causing the mass flux to vary. At the same time, however, the incident vorticity will induce the shedding of additional vorticity from the edges of the aperture. The local unsteady flow is therefore characterized by the presence of two distributions of unsteady vorticity – that due to the incident vortex pair and that shed from the wall in response to excitation by the vortex pair. Both of these behave as sound sources, inasmuch as they both influence the unsteady flow through the aperture. Problems of this type are not normally amenable to analytical investigation. However, it is clearly of great value to have available even an approximate solution, and this can be achieved if the problem is linearized by assuming that the unsteady local motion of the jet produced by the vortex pair (with circulations $\pm\Gamma$) is well approximated by linearizing with respect to the vortex strength Γ. The analytical procedure is an extension of that used in [7–9] to model the interaction of turbulence with pressure-release cavity flows.

The hydroacoustic problem is formulated in Section 2. The hydrodynamic, near field properties of the unsteady flow are determined according to linear theory with a Kutta condition applied in Section 3 and the sound radiated from the aperture is then calculated (Section 4).

2. The hydroacoustic problem

Consider the modified form of the aperture flow depicted in Figure 2, in which a vortex pair is incident symmetrically on the aperture from $x_2 > 0$, where the coordinate origin is taken at the centre of the aperture and where the flow may be regarded as taking place in the z-plane ($z = x_1 + \mathrm{i}x_2$). Free streamlines separate smoothly from the edges $z = \pm h$ of the aperture forming the boundaries of the jet. The pressure p may be taken to vanish in the cavity region outside the jet and the influence of gravity is neglected, at least in the flow close to the aperture. Bernoulli's equation therefore implies that the flow speed is constant along the mean free streamlines and equal to the asymptotic jet velocity U.

The core of each vortex is of infinitesimal cross-section and located at $z_o^{\pm}(t) = \pm x_{1o}(t) + \mathrm{i}x_{2o}(t)$ at time t, and the flow is assumed to be linearly disturbed from the steady state by the vortex pair. In this approximation the self-induced velocity of the vortex pair is required to be negligible compared to the mean flow velocity, and the influence on the motion of image vortices in the boundaries is also negligible. Then each vortex convects along a streamline of the undisturbed flow and is swept through the aperture at the local mean flow velocity $\mathbf{v}_o(z_o^{\pm}(t))$, say.

The sound produced as the vortex pair is accelerated into the jet is governed by Lighthill's acoustic analogy [5], [6, Chapter 2]. In aqueous flows of this type the mean-flow Mach number is infinitesimal and the convection of sound by the mean-flow can therefore be neglected. Lighthill's equation of aerodynamic sound may then be taken in the form [6, Section 2.3.3]

$$\left(\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)B = \mathrm{div}(\boldsymbol{\omega} \wedge \mathbf{v}), \tag{1}$$

where $B = p/\rho_o + \frac{1}{2}v^2$, and where ρ_o, c_o are, respectively, the mean density and sound speed in the water, $\mathbf{v} \equiv \mathbf{v}(\mathbf{x}, t)$ the flow velocity (including that of the mean flow), and $\boldsymbol{\omega} = \mathrm{curl}\,\mathbf{v}$ is the vorticity. Bernoulli's equation implies that $B \equiv -\partial\varphi_B/\partial t + \text{constant}$ in those regions where the motion is irrotational and described by a velocity potential φ_B. It follows that we may take $B = 0$ for the undisturbed flow. Viscous dissipation is ignored within the body of the fluid, but its effect at solid boundaries on both the mean and unsteady components of the flow is included by application of the Kutta condition at the aperture edges [6, Section 5.3], [10].

When the source term on the right of Equation (1) is known, the solution $B(\mathbf{x}, t)$ determines the sound produced during the interaction of the vortex pair with the aperture and jet. The solution is required to satisfy the condition of vanishing normal component of velocity on the upper walls $|x_1| > h$, $x_2 = +0$, and (in the linearized approximation) the condition of vanishing pressure on the free streamlines defining the undisturbed boundary $\mathrm{S_J}$, say, of the jet.

Equation (1) will be solved by introducing a Green's function $G(\mathbf{x},\mathbf{y},t-\tau)$, which is a solution of (1) with outgoing-wave behaviour when the right-hand side is replaced by the impulsive point source $\delta(\mathbf{x}-\mathbf{y})\delta(t-\tau)$. Then, by the usual procedure (see, *e.g.*, [6, Section 2.2]), in the high-Reynolds-number approximation, the solution of Equation (1) can be cast in the form

$$B(\mathbf{x},t) = -\int (\boldsymbol{\omega}\wedge\mathbf{v})(\mathbf{y},\tau)\cdot\frac{\partial G}{\partial \mathbf{y}}(\mathbf{x},\mathbf{y},t-\tau)\,\mathrm{d}^2\mathbf{y}\,\mathrm{d}\tau + \int_{\mathrm{S}}\left(B(\mathbf{y},\tau)\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y},t-\tau)+G(\mathbf{x},\mathbf{y},t-\tau)\frac{\partial v_n}{\partial \tau}(\mathbf{y},\tau)\right)\mathrm{d}s(\mathbf{y})\,\mathrm{d}\tau. \tag{2}$$

The integrations with respect to τ are over all values of the retarded time $(-\infty,\ \infty)$, and $s(\mathbf{y})$ is distance along the boundary $\mathrm{S}=\mathrm{S_W}+\mathrm{S_J}$, which consists, respectively, of the upper side $\mathrm{S_W}$ of the rigid wall $(y_2=+0)$, and the mean free streamlines $\mathrm{S_J}$ of the jet. The first integral is over the fluid region where $\boldsymbol{\omega}\neq\mathbf{0}$, and is therefore confined to the vortex pair. In the second integral over S, y_n is distance measured normal to S and directed *into* the fluid and v_n is the normal component of velocity on S.

Equation (2) is simplified by choosing the Green's function to satisfy the conditions

$$G(\mathbf{x},\mathbf{y},t-\tau)=0 \quad \text{for } \mathbf{y} \text{ on } \mathrm{S_J}, \qquad \frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y},t-\tau)=0 \quad \text{for } \mathbf{y} \text{ on } \mathrm{S_W}. \tag{3}$$

Then, because $\partial v_n/\partial\tau=0$ on the rigid wall $\mathrm{S_W}$, Equation (2) reduces to

$$B(\mathbf{x},t) = -\int (\boldsymbol{\omega}\wedge\mathbf{v})(\mathbf{y},\tau)\cdot\frac{\partial G}{\partial \mathbf{y}}(\mathbf{x},\mathbf{y},t-\tau)\,\mathrm{d}^2\mathbf{y}\,\mathrm{d}\tau + \int_{\mathrm{S_J}} B(s,\tau)\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y},t-\tau)\,\mathrm{d}s\,\mathrm{d}\tau. \tag{4}$$

Now $\boldsymbol{\omega}=\Gamma\mathbf{k}\{\delta(\mathbf{y}-\mathbf{x}_o^+(\tau))-\delta(\mathbf{y}-\mathbf{x}_o^-(\tau))\}$, where $\mathbf{x}_o^\pm(\tau)=(\pm x_{1o}(\tau), x_{2o}(\tau))$ and $\mathbf{v}(\mathbf{x}_o^\pm(\tau),\tau)\equiv \mathbf{v}_o(\mathbf{x}_o^\pm(\tau))$ in the linearized approximation, where $\mathbf{k}$ is a unit vector out of the plane of the paper in Figure 2. Therefore, Equation (4) becomes

$$B(\mathbf{x},t) = -\Gamma\int v_o(\mathbf{x}_o^+(\tau))\frac{\partial G}{\partial y_\perp}(\mathbf{x},\mathbf{x}_o^+(\tau),t-\tau)\,\mathrm{d}\tau + \Gamma\int v_o(\mathbf{x}_o^-(\tau))\frac{\partial G}{\partial y_\perp}(\mathbf{x},\mathbf{x}_o^-(\tau),t-\tau)\,\mathrm{d}\tau + \int_{\mathrm{S_J}} B(s,\tau)\frac{\partial G}{\partial y_n}(\mathbf{x},\mathbf{y},t-\tau)\,\mathrm{d}s\,\mathrm{d}\tau, \tag{5}$$

where, for each vortex $\pm\Gamma$, the local coordinate $y_\perp$ is measured along the normal to the streamline on which it convects and *to the left* of the direction of the mean flow. The first two integrals can be performed as soon as the functional form of Green's function is known. To evaluate the integral over the free streamlines $\mathrm{S_J}$ it is first necessary to determine $B(s,t)$ on $\mathrm{S_J}$.

3. Interaction of the vortex pair and the jet

At low Mach numbers $M=U/c_o$ the sound generated during the passage of the vortex pair into the jet has wavelength $\sim h/M$ $(\gg h)$ that greatly exceeds the size of the interaction region. The local motion in the vicinity of the aperture is then identical with that for an incompressible fluid. When the motion is only linearly disturbed (by the vortices) from the steady state, it can be calculated by extension of the method of conformal transformation traditionally used to analyse free-streamline problems [1, Section 73], [2, Section 6.13], [3, 4]. To do this, the undisturbed region occupied by the mean flow in the z-plane is mapped onto the upper half of the ζ-plane (Figure 3).

3.1. The mean flow

Let the boundary ABCC′B′A′ of the undisturbed mean flow of Figure 1 map onto the real ζ-axis with the correspondences shown in Figure 3a, with the edges B, B′ ($z=\mp h$) and the far jet CC′ ($z=-\mathrm{i}\infty$) mapping, respectively, into the points $\zeta=\mp 1$ and $\zeta=0$. Then the mean flow in the ζ-plane may be attributed to a sink of strength $2\sigma hU$ at the origin, and the complex potential $w=\phi+\mathrm{i}\psi$ of this motion and the transformation $z=z(\zeta)$ are [1, Section 73], [2, Section 6.13], [3, Section II.5], [4, Section II.C]

$$\left.\begin{aligned} w &= -\tfrac{2\sigma hU}{\pi}\log\zeta + 2\mathrm{i}\sigma hU, \qquad \sigma=\pi/(2+\pi) \\ z &= \sigma h\left[1+\tfrac{2}{\pi}\left(\zeta+\sqrt{\zeta^2-1}\right)-\tfrac{2\mathrm{i}}{\pi}\left\{\log\left(1-\mathrm{i}\sqrt{\zeta^2-1}\right)-\log\zeta\right\}\right], \end{aligned}\right\}\quad \eta\geq 0, \tag{6}$$

where $\zeta=\xi+\mathrm{i}\eta$ (ξ, η being the real and imaginary parts of ζ) and the principal values of the square roots and the logarithm are to be taken.

The mean flow occupies the region $-\infty<\phi<+\infty$, $0<\psi<2\sigma hU$ of the w-plane (Figure 3b). The mean streamlines $\psi(x_1,x_2)=$ constant map into the rays

$$\zeta=-\,\mathrm{e}^{-\pi w/2\sigma hU} \equiv -\,\mathrm{e}^{-\frac{\pi}{2\sigma hU}(\phi+\mathrm{i}\psi)}, \quad -\infty<\phi<+\infty \tag{7}$$

converging on the origin in the ζ-plane. We shall assume that the *left hand* vortex $-\Gamma$ at $z=z_o^-(t)$ travels along the mean streamline $\psi=\psi_o$, and is at $w\equiv w_o=\phi_o(t)+\mathrm{i}\psi_o$ at time t, where the time is measured from the instant that the vortex crosses the real z-axis (*i.e.*, $\phi=0$). Then the right vortex travels along the streamline $\psi=2\sigma hU-\psi_o$, as indicated in the figure.

3.2. Motion of the vortex pair

In the linearized approximation the complex positions $z_o^\pm=\pm x_{1o}+\mathrm{i}x_{2o}$ of the vortices at time t are determined by the equations of motion $\mathrm{d}z_o^\pm/\mathrm{d}t=(\mathrm{d}w/\mathrm{d}z)^*_{z=z_o^\pm}$, where the asterisk denotes complex conjugate. These equations must be solved numerically, and it is convenient to do this by first determining the corresponding *rectilinear* motions in the w-plane of Figure 3b, where the vortices at z_o^-, z_o^+ travel, respectively, along the straight lines $\psi=\psi_o$, $2\sigma hU-\psi_o$ for a prescribed constant value of ψ_o. Because of the symmetry of the problem, it is obviously sufficient to consider only the motion of, say, the left vortex at $z=z_o(t)\equiv z_o^-(t)$, and henceforth the superscript '−' will be dropped.

Then, setting $w(z_o)=\phi_o+\mathrm{i}\psi_o$, it follows from Equations (6), (7) that

$$\left.\begin{aligned} \frac{\mathrm{d}\phi_o}{\mathrm{d}t} &= |w_o'|^2 \equiv U^2\Big/\left|\zeta_o+\sqrt{\zeta_o^2-1}\right|^2, \\ \text{where}\quad \zeta_o &= -\,\mathrm{e}^{-\frac{\pi}{2\sigma hU}(\phi_o+\mathrm{i}\psi_o)}. \end{aligned}\right\} \tag{8}$$

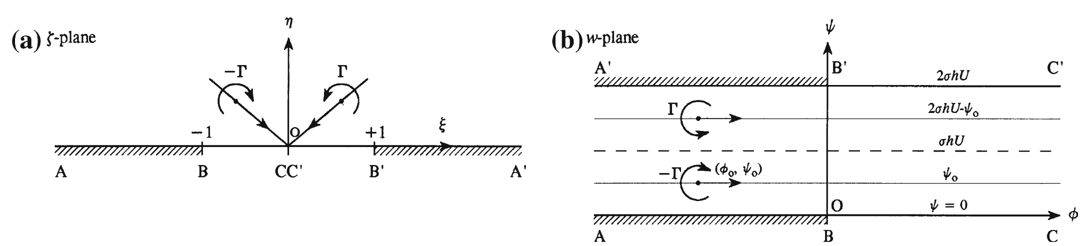

Figure 3. Image of the hydrodynamic flow (a) in the upper half of the ζ-plane, (b) in the w-plane. The cross-hatched sections of the boundaries correspond to the wetted surfaces AB and B′A′ of Figure 1.

Here we have introduced the shorthand notations $w' \equiv w'(z) = \mathrm{d}w/\mathrm{d}z$ evaluated at a general point z, and $w'_o = w'(z_o)$. Equations (8) determine the position of ζ_o on the streamline $\psi_o =$ constant as a function of the nondimensional time Ut/h. The corresponding position z_o in the physical plane is found from the second of Equations (6). The value of ψ_o and the time origin can be adjusted to ensure that the vortices cross the real axis in the physical plane at $x_1 = \pm x_\Gamma$ $(0 < x_\Gamma < h)$ at $t = 0$ for a prescribed value of x_Γ.

The paths plotted in Figure 2 correspond to $x_\Gamma = 0{\cdot}375h$. The points marked on the curves represent the positions of the vortices at different values of Ut/h. The progressively increasing spacings of successive points on these streamlines illustrate how the vortices are accelerated into the jet; this is followed by the rapid attainment of convection at the constant jet speed U.

3.3. The hydrodynamic approximation

In the absence of the vortex pair $B \equiv -\partial\varphi_B/\partial t = 0$. By hypothesis $B(\mathbf{x}, t)$ is linearly perturbed about this value when the vortex pair is swallowed by the aperture flow. Incompressible *hydrodynamics* dominate the motion close to and within the jet, and can be investigated using Lighthill's equation (1) by discarding the term involving the sound speed c_o (taking the formal limit $c_o \to \infty$). A knowledge of the solution $B(\mathbf{x}, t)$ is required only on the free streamlines $\mathrm{S_J}$ in order to evaluate the final integral in Equation (5).

The solution is facilitated by working in the w-plane of Figure 3b. However, the motion is obviously symmetrical with respect to the central streamline $\psi = \sigma hU$ (the broken line in Figure 3b), and it is actually sufficient to solve the incompressible problem within the smaller region $-\infty < \phi < \infty$, $0 < \psi < \sigma hU$ with the mean streamline $\psi = \sigma hU$ replaced by a rigid wall on which $\partial B/\partial\psi = 0$. Only the image $w_o(t) = \phi_o(t) + \mathrm{i}\psi_o$ of the left-hand vortex at z_o, *i.e.*, at $\mathbf{x}_o(t) = (-x_{1o}(t), x_{2o}(t))$, lies in this region, for which

$$\delta(\mathbf{x} - \mathbf{x}_o) = |w'_o|^2 \delta(\phi - \phi_o)\delta(\psi - \psi_o). \tag{9}$$

It follows for this vortex, using the Cauchy-Riemann equations, that

$$\mathrm{div}(\boldsymbol{\omega} \wedge \mathbf{v}) \ = \ -\Gamma |w'_o|^2 |w'|^2 \delta(\phi - \phi_o)\delta'(\psi - \psi_o), \tag{10}$$

where the prime on the δ-function denotes differentiation with respect to the argument.

The identity

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = |w'|^2 \left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2} \right)$$

now permits the incompressible form of Equation (1) to be put in the reduced form

$$\left(\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2} \right) B \ = \ \Gamma |w'_o|^2 \delta(\phi - \phi_o)\delta'(\psi - \psi_o), \quad \text{for} \quad 0 < \psi < \sigma hU. \tag{11}$$

In addition to the condition $\partial B/\partial\psi = 0$ on $\psi = \sigma hU$, the solution of Equation (11) must also have vanishing normal derivative $\partial B/\partial\psi = 0$ on the negative real axis $\phi < 0$, $\psi = +0$, the image of the rigid wall AB of Figure 1.

The condition to be imposed on the positive real axis ($\phi > 0$, $\psi = +0$: the image of the mean free streamline BC) is obtained by considering the perturbation Bernoulli equation

$$\frac{\partial \varphi_B}{\partial t} + \frac{\delta p}{\rho_o} + \mathbf{v}_o \cdot \nabla \varphi_B = 0,$$

where $\mathbf{v}_o(\mathbf{x})$ is the mean flow velocity and δp and φ_B denote, respectively, the perturbation pressure and velocity potential induced by the passage of the vortex pair. This equation remains valid on a fixed control surface close to the perturbed free streamline, and δp and φ_B must differ from their respective free-streamline values by terms of second order. Hence, because $\delta p = 0$ on the perturbed free streamline, the potential φ_B satisfies to first order $\partial\varphi_B/\partial t + U\partial\varphi_B/\partial s = 0$ on BC, where $v_o = U$ and s is taken to increase from zero at B along the streamline. Thus, because $\mathrm{d}s = \mathrm{d}\phi/U$ on BC, the required condition is that $\partial B/\partial t + U^2\partial B/\partial\phi = 0$ on $\phi > 0$, $\psi = +0$

Equation (11) may be solved by putting

$$B = B_i + B_s \tag{12}$$

where $B_i(\phi, \psi, t)$ is the particular solution that satisfies $\partial B_i/\partial\psi = 0$ on the *entire* lower and upper boundaries: $\psi = 0$, σhU $(-\infty < \phi < \infty)$. The remaining component B_s will then be the solution of the boundary-value problem

$$\left(\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\psi^2}\right)B_s = 0, \qquad -\infty < \phi < \infty, \quad 0 < \psi < \sigma hU,$$

$$\text{where} \quad \frac{\partial B_s}{\partial\psi} = 0 \qquad \text{on} \begin{cases} \psi = 0, \phi < 0, \\ \psi = \sigma hU, -\infty < \phi < \infty, \end{cases} \tag{13}$$

$$\text{and} \quad \left(\frac{\partial}{\partial t} + U^2\frac{\partial}{\partial\phi}\right)(B_i + B_s) = 0 \quad \text{on} \quad \psi = 0, \ \phi > 0. \tag{14}$$

In Equation (11) $\Gamma|w_o'|^2$ is known as a function of the current position (ϕ_o, ψ_o) of the left-hand vortex in the w-plane, and accordingly as a function of the time. The particular solution B_i may therefore be conveniently derived by Fourier transformation with respect to the dependence on ϕ. Equations (4) and (14) indicate that the value of B_i is required only on the mean streamline BC ($\psi = 0$, $\phi > 0$), on which it is easily shown that

$$B_i(\phi, 0, t) = \frac{\Gamma|w_o'|^2}{2\pi}\int_{-\infty}^{\infty}\frac{\sinh\{K(\psi_o - \sigma hU)\}}{\sinh(K\sigma hU)}\,\mathrm{e}^{\mathrm{i}K(\phi - \phi_o(t))}\,\mathrm{d}K. \tag{15}$$

To use this in condition (14) to determine B_s, we introduce the Fourier time transform $\mathcal{F}(K, \omega)$ of the time-dependent terms on the right of (15):

$$\mathcal{F}(K, \omega) = \frac{\Gamma}{2\pi}\int_{-\infty}^{\infty}|w_o'|^2\,\mathrm{e}^{\mathrm{i}(\omega t - K\phi_o(t))}\,\mathrm{d}t, \tag{16}$$

so that

$$B_i(\phi, 0, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{F}(K, \omega)\sinh\{K(\psi_o - \sigma hU)\}}{\sinh(K\sigma hU)}\,\mathrm{e}^{\mathrm{i}(K\phi - \omega t)}\,\mathrm{d}K\,\mathrm{d}\omega. \tag{17}$$

We can now determine $B_s(\phi, \psi, t)$ by first obtaining the solution $\hat{B}_s$, say, for the simpler case where B_i in condition (14) is replaced by $\mathrm{e}^{\mathrm{i}K\phi}$ and $\partial/\partial t$ by $-\mathrm{i}\omega$, followed by application of the integral operator

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{F}(K, \omega)\sinh\{K(\psi_o - \sigma hU)\}}{\sinh(K\sigma hU)}\Big(\cdot\Big)\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}K\,\mathrm{d}\omega. \tag{18}$$

For this reduced problem condition (14) becomes

$$\hat{B}_s + e^{iK\phi} = \alpha e^{i\nu\phi} \quad (\psi = 0,\ \phi > 0), \quad \nu = \omega/U^2, \tag{19}$$

where α is a constant to be determined.

We may now put

$$\hat{B}_s(\phi, \psi) = \int_{-\infty}^{\infty} \mathcal{A}(k) \cosh\{k(\psi - \sigma hU)\} e^{ik\phi}\, dk. \tag{20}$$

This is a solution of Laplace's equation that satisfies the second of conditions (13). The first of conditions (13) and condition (19) are also satisfied provided $\mathcal{A}(k)$ is a solution of the dual integral equations

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} k\mathcal{A}(k) \sinh(k\sigma hU) e^{ik\phi}\, dk = 0, && \phi < 0, \\ &\int_{-\infty}^{\infty} \mathcal{A}(k) \cosh(k\sigma hU) e^{ik\phi}\, dk \ + \ e^{iK\phi} \ - \ \alpha e^{i\nu\phi} = 0, && \phi > 0. \end{aligned}\right\} \tag{21}$$

These equations are readily solved by routine application of the *Wiener-Hopf* procedure, and the reader is referred to [11, Chapter 1], [12, Chapter 4] for details. We seek the particular solution that accords with the Kutta condition that the unsteady flow leaves the edge B smoothly. This condition determines the value of α by requiring that $\partial \hat{B}_s/\partial\psi \to 0$ at B, and yields

$$\alpha = \left(\frac{K + i0}{\nu + i0}\right) \frac{F_+(\nu)}{F_-(K)}, \tag{22}$$

where the functions $F_\pm(k)$ are, respectively, regular and non-zero in the half-planes $\mathfrak{Im}\, k \gtrless 0$, and represent the multiplicative decomposition of $F(k) \equiv F_+(k)F_-(k) = k\sigma hU \coth(k\sigma hU)$, *viz.*

$$F_\pm(k) = \frac{\sqrt{\pi}\,\Gamma\left(1 \mp \frac{ik\sigma hU}{\pi}\right)}{\Gamma\left(\frac{1}{2} \mp \frac{ik\sigma hU}{\pi}\right)}, \tag{23}$$

where here Γ denotes the Γ-function (see [11, p. 41, Ex. 1.11]).

It is unnecessary to proceed to any further discussion of details of the solution of the incompressible problem, because the result (22) and Equation (19) determine completely the behaviour of B on the mean streamline BC for the component $e^{iK\phi}$ of B_i. Hence, applying the operator (18), we find

$$B(\phi, 0, t) \ = \ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{F}(K,\omega) \sinh\{K(\psi_o - \sigma hU)\}}{\sinh(K\sigma hU)} \left(\frac{K + i0}{\nu + i0}\right) \frac{F_+(\nu)}{F_-(K)} e^{-i(\omega t - \nu\phi)}\, dK\, d\omega, \ \ \phi > 0. \tag{24}$$

In this formula $\mathcal{F}(K,\omega)$ is defined by (16) and $F_+(\nu)$, $F_+(K)$ by (23). It is clear by inspection that the only singularities in the complex K- and ω-planes are simple poles, and therefore that the integrations with respect to these variables can be effected by residues. By recalling that $\phi = Us$ on BC, where s is distance measured along the free streamline in the physical plane (so that $B(\phi, 0, t) \equiv B(s, t)$ in the notation of Equation (5)), it follows that on BC,

$$B(s,t)=\frac{-\mathrm{i}\Gamma}{(2\pi hU)^2}\int_{-\infty}^{\infty}|w_o'|^2\,\mathfrak{Im}\left(\frac{\zeta_o}{(\zeta_o^2-1)^{\frac{3}{2}}}\right)\frac{F_+(\nu)}{(\nu+\mathrm{i}0)}\,\mathrm{e}^{-\mathrm{i}\omega(t-\tau-s/U)}\,\mathrm{d}\omega\,\,\mathrm{d}\tau, \tag{25}$$

$$=\frac{-\pi\Gamma}{2(\sigma h)^2}\int_{-\infty}^{t-s/U}|w_o'|^2\,\mathfrak{Im}\left(\frac{\zeta_o}{(\zeta_o^2-1)^{\frac{3}{2}}}\right)\frac{\mathrm{d}\tau}{\left(1-\mathrm{e}^{-\frac{\pi U}{\sigma h}(t-\tau-s/U)}\right)^{\frac{1}{2}}}, \tag{26}$$

where w_o' and ζ_o are evaluated at the position of the left vortex at time τ, and the second of Equations (8) has been used to express $\phi_o+\mathrm{i}\psi_o$ in terms of ζ_o. Both of the representations (25), (26) are used below.

3.4. Bound vorticity on the free streamlines

The flow in the jet may be regarded as 'slipping' over the undisturbed free streamlines BC, B′C′. The limiting tangential component of velocity $\partial\varphi_B/\partial s$ determines the component of the slip velocity produced by the vortex pair and vortex 'shedding' from the edges of the aperture, *i.e.*, the unsteady *bound vorticity*. Thus, on BC $B(s,t)=-\partial\varphi_B/\partial t\equiv U\partial\varphi_B/\partial s$, where $\gamma\equiv\gamma(t-s/U)=-\partial\varphi_B/\partial s$ is the perturbation circulation per unit length of the free streamline (taken to be positive with respect to rotation about the **k**-direction, out of the plane of the paper in Figures 1 and 2). It convects as a frozen pattern on BC at the free streamline velocity U. The contribution to the acoustic radiation from the free-streamline integral of Equation (5) is therefore similar to the direct radiation from the vortex pair, except that the vorticity is distributed on the free streamlines.

The total unsteady bound vorticity Γ_s on BC at time t is

$$\Gamma_s\;=\;-\int_0^{\infty}\frac{\partial\varphi_B}{\partial s}(s,t)\,\mathrm{d}s. \tag{27}$$

Using the representation (26) we can write

$$\frac{\Gamma_s(t)}{\Gamma}=\frac{\pi}{2h\sigma^2}\int_0^{\infty}\mathrm{d}s\int_{-\infty}^{t-s/U}\frac{\mathfrak{Im}\left\{\zeta_o/(\zeta_o^2-1)^{\frac{3}{2}}\right\}}{|\zeta_o+(\zeta_o^2-1)^{\frac{1}{2}}|^2}\frac{\mathrm{d}\tau}{\left(1-\mathrm{e}^{-\frac{\pi U}{\sigma h}(t-\tau-s/U)}\right)^{\frac{1}{2}}}, \tag{28}$$

This is plotted in Figure 4a as a function of Ut/h for several values of x_Γ/h, where in all cases the vortex pair is assumed to cut the plane of the aperture at $t=0$. When x_Γ/h is small, the vortices in the vortex pair are close together, their influence on the jet becomes small and $\Gamma_s\ll\Gamma$. In the alternative limit $x_\Gamma/h\to 1$ each vortex follows a trajectory immediately adjacent to the wetted surfaces of the walls AB and A′B′ and is subsequently convected along the corresponding free stream line. In this case the Kutta condition requires that $\Gamma_s=\Gamma$ and there is no disturbed motion. At intermediate values of x_Γ/h the bound vorticity grows slowly prior to the arrival of the vortex pair at the aperture at $t=0$, when it jumps to a maximum value that varies rapidly with x_Γ/h and peaks near $x_\Gamma/h=0{\cdot}6$, as indicated in Figure 4b.

If the Kutta condition is not applied (so that $B_s\equiv 0$) there would still be a distribution of bound vorticity on the free streamlines, determined by replacing the 'exact' value of $\partial\varphi_B/\partial s$ in (27) by $\partial\varphi_B/\partial s=-B_i(\phi,0,t)/U$ given by the 'hard wall' component (15). The total circulation in this case Γ_i, say, when Ut/h is large, it is readily shown to be

$$\frac{\Gamma_i}{\Gamma}=1-\frac{\psi_o}{\sigma hU}. \tag{29}$$

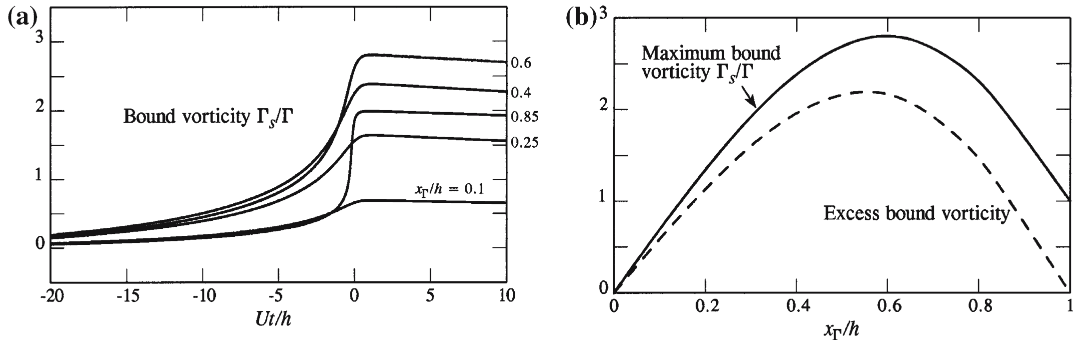

Figure 4. (a) Growth of the relative bound vorticity Γ_s/Γ on the free streamline BC for $x_\Gamma/h = 0{\cdot}1,\ 0{\cdot}25,\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}85$; (b) maximum relative bound vorticity and excess bound vorticity as functions of x_Γ/h.

The *excess* bound vorticity shed from the aperture edge in order to satisfy the Kutta condition is obtained by taking $\partial\varphi_B/\partial s = -B_s(\phi, 0, t)/U$. The peak value of the excess bound vorticity is given in a first approximation by subtracting Γ_i determined by (29) from the peak value of Γ_s. The result is plotted as the broken line curve in Figure 4b. This curve shows that the Kutta condition results in the shedding of substantial excess vorticity, indicating that this can be a particularly significant acoustic source.

The circulation density $\gamma(t - s/U)$ tends to be confined to a relatively short interval in s propagating at speed U along the free streamline. This is illustrated in Figure 5, where $\gamma(t - s/U)h/\Gamma$ is plotted as a function of $(Ut - s)/h$ for $x_\Gamma/h = 0{\cdot}5$ and $0{\cdot}75$.

4. The acoustic radiation

4.1. Green's function

At very small Mach numbers $M = U/c_o$ the wavelength of the sound produced as the vortex pair encounters the aperture and jet has characteristic wavelength $\sim h/M \gg h$. The motion in the interaction region is effectively the same as for an incompressible fluid, whereas the acoustic waves becomes dominant in the upper region of Figure 1 only at large distances $|\mathbf{x}| \gg h$ from the aperture. Lighthill's equation (1) and the integral representation (5) determine the sound at these large distances in terms of the calculated near field. This simplifies the evaluation of (5) because it is then only necessary to know the Green's function $G(\mathbf{x}, \mathbf{y}, t - \tau)$ for $|\mathbf{x}| \gg h$ and for source points $\mathbf{y} \sim O(h)$ within an acoustically *compact* neighbourhood of the aperture.

To determine this compact approximation for G we first set

$$G(\mathbf{x}, \mathbf{y}, t - \tau) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(\mathbf{x}, \mathbf{y}, \omega)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{30}$$

so that $\hat{G}(\mathbf{x}, \mathbf{y}, \omega)$ satisfies

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k_o^2\right)\hat{G} = \delta(\mathbf{x} - \mathbf{y}), \tag{31}$$

where $k_o = \omega/c_o$ is the acoustic wavenumber. The compact approximation implies that $k_o h \ll 1$.

The reciprocal theorem [6, Section 1.9], [13, p. 145] permits the solution to be derived *as a function of* $\mathbf{y}$ when the point source on the right of (31) is regarded as situated at a point $\mathbf{x}$

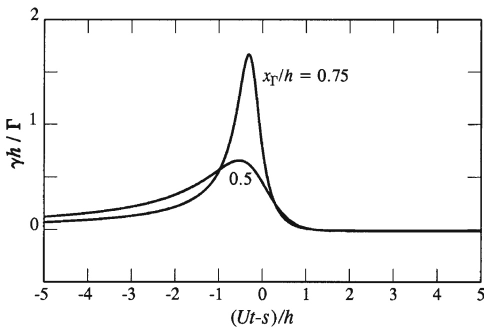

Figure 5. Distribution of circulation density $\gamma(t - s/U)$ on the free streamline BC for $x_\Gamma/h = 0{\cdot}5,\ 0{\cdot}75$.

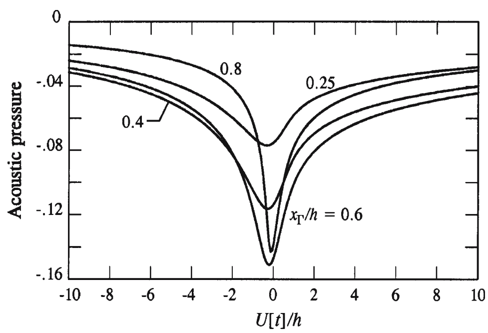

Figure 7. The nondimensional, far field acoustic pressure $(p_\Gamma + p_S) \Big/ \left(\dfrac{\rho_o U \Gamma}{h M^{\frac{1}{2}}}\right)\left(\dfrac{h}{|\mathbf{x}|}\right)^{\frac{1}{2}}$ plotted as a function of the retarded time $[t] = t - |\mathbf{x}|/c_o$ for $M = 0{\cdot}01$ and $x_\Gamma/h = 0{\cdot}25,\ 0{\cdot}4,\ 0{\cdot}6,\ 0{\cdot}8$.

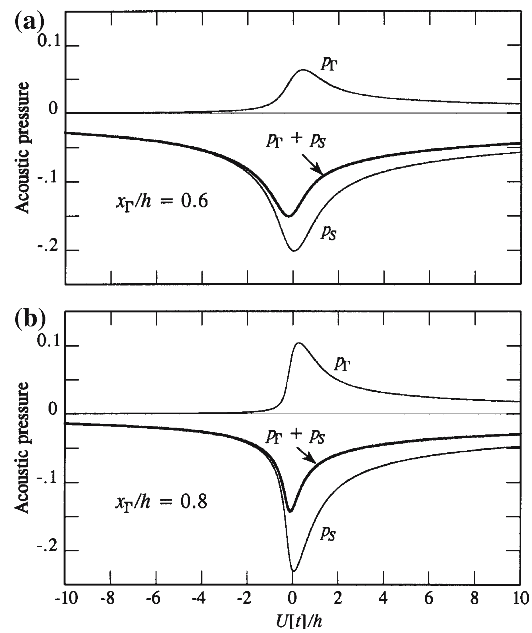

Figure 6. The nondimensional, far field acoustic pressures (p_Γ, p_S and $p_\Gamma + p_S$) $\Big/ \left(\dfrac{\rho_o U \Gamma}{h M^{\frac{1}{2}}}\right)\left(\dfrac{h}{|\mathbf{x}|}\right)^{\frac{1}{2}}$ plotted as functions of the retarded time $[t] = t - |\mathbf{x}|/c_o$ for $M = 0{\cdot}01$ and (a) $x_\Gamma/h = 0{\cdot}6$, (b) $x_\Gamma/h = 0{\cdot}8$.

in the far field in the upper region of Figure 1. The procedure is very familiar in problems of this type (*e.g.*, see [14]) and only an abbreviated outline will be necessary. Thus, in the water ($y_2 > 0$) at large distances $|\mathbf{y}| \gg h$ from the aperture

$$\hat{G}(\mathbf{x}, \mathbf{y}, \omega) = -\ c_1 \frac{\mathrm{i}}{4} \mathrm{H}_0^{(1)}(k_o|\mathbf{y}|) - \frac{\mathrm{i}}{4}\left\{\mathrm{H}_0^{(1)}(k_o|\mathbf{y} - \mathbf{x}|) + \mathrm{H}_0^{(1)}(k_o|\mathbf{y} - \bar{\mathbf{x}}|)\right\}, \tag{32}$$

where $\mathrm{H}_0^{(1)}$ is a Hankel function and $\bar{\mathbf{x}} = (x_1, -x_2)$ is the image of $\mathbf{x}$ in the plane of ABB′A′. The first term on the right is the field of a two-dimensional acoustic source of strength $c_1 \equiv c_1(\mathbf{x})$ per unit length distributed along the centreline of the aperture; it represents the effect of unsteady flow into and out of the aperture produced by the incident field of the unit source of Equation (31) at $\mathbf{x}$ (given by the first term in the brace brackets of (32)) and its image at $\bar{\mathbf{x}}$ (given by the second such term).

Within a compact neighbourhood of the aperture, where $|\mathbf{y}| \ll |\mathbf{x}|$ and $k_o|\mathbf{y}| \ll 1$, we can write [15, Chapter 9]

$$\hat{G}(\mathbf{x}, \mathbf{y}, \omega) \approx -c_1 \frac{\mathrm{i}}{4}\left(1 + \frac{2\mathrm{i}}{\pi}\log\left[\frac{\varpi k_o|\mathbf{y}|}{2}\right]\right) - \frac{\mathrm{i}}{2}\mathrm{H}_0^{(1)}(k_o|\mathbf{x}|), \quad \varpi = 1{\cdot}781072. \tag{33}$$

The motion at the aperture and in the jet (where $\mathbf{y} \sim O(h)$) may be regarded as incompressible, and represented by the following solution of Laplace's equation

$$\hat{G}(\mathbf{x}, \mathbf{y}, \omega) \approx -c_2 \frac{\mathrm{i}}{4} \Re\mathfrak{e} \left\{ \log \left(\zeta + \sqrt{\zeta^2 - 1} \right) \right\}, \quad \zeta = \zeta(z), \quad z = y_1 + \mathrm{i} y_2, \tag{34}$$

where $c_2 \equiv c_2(\mathbf{x})$ remains to be determined and $\zeta(z)$ is defined by the second of Equations (6). This tends to the field of a monopole source for $|\mathbf{y}| \gg h$ in $y_2 > 0$ and satisfies $\hat{G} = 0$ when $\mathbf{y}$ lies on the free streamlines BC and B′C′ (on the interval $-1 < \xi < 1$ of the real ζ-axis where $\log\left(\zeta + \sqrt{\zeta^2 - 1}\right)$ is pure imaginary), and $\partial\hat{G}/\partial y_2 = 0$ on AB, B′A′, where $|\xi| > 1$. The limiting value of this formula in the 'overlap' region $h \ll |\mathbf{y}| \ll 1/k_o$ is found from the second of Equations (6) to be

$$\hat{G}(\mathbf{x}, \mathbf{y}, \omega) \approx -c_2 \frac{\mathrm{i}}{4} \log \left(\frac{\pi |\mathbf{y}|}{2\sigma h} \right). \tag{35}$$

This is compatible with Equation (33) provided $c_1 = -\mathrm{i}\pi c_2/2$, and

$$c_2 = \frac{-4\mathrm{i}\mathrm{H}_0^{(1)}(k_o|\mathbf{x}|)}{\pi + 2\mathrm{i}\log(\varpi\sigma k_o h/\pi)}. \tag{36}$$

By inserting this result into Equation (34) it then follows from (30) that the desired compact Green's function can be expressed in the form

$$G(\mathbf{x}, \mathbf{y}, t - \tau) \approx -\left(\frac{2c_o}{\pi^3 |\mathbf{x}|} \right)^{\frac{1}{2}} \frac{\chi(t - \tau - |\mathbf{x}|/c_o)}{\sqrt{t - \tau - |\mathbf{x}|/c_o}} \Phi(\mathbf{y}), \qquad |\mathbf{x}| \to \infty,$$

$$\chi(t) = \mathrm{H}(t) \int_0^\infty \frac{\log(\varpi\sigma h \mu^2/\pi c_o t) \mathrm{e}^{-\mu^2} \, \mathrm{d}\mu}{[\log(\varpi\sigma h\mu^2/\pi c_o t)]^2 + \pi^2}, \qquad \varpi = 1{\cdot}781072,$$

$$\Phi(\mathbf{y}) = \Re\mathfrak{e} \left\{ \log \left(\zeta + \sqrt{\zeta^2 - 1} \right) \right\}, \quad \zeta = \zeta(z), \qquad z = y_1 + \mathrm{i} y_2. \tag{37}$$

Equations (37) describe an *omnidirectional* acoustic wave spreading out cylindrically in the upper region from a source at $\mathbf{y}$ near the aperture or within the jet, whose amplitude decreases like $1/\sqrt{|\mathbf{x}|}$ at large distances (as appropriate in two dimensions).

4.2. Direct radiation from the vortex

At infinitesimal Mach number and at large distances from the aperture, the acoustic component of $B(\mathbf{x}, t) \approx p(\mathbf{x}, t)/\rho_o$, where $p(\mathbf{x}, t)$ is the acoustic pressure perturbation. The first two integrals in Equation (5) represent the contribution p_Γ, say, from the *direct* interaction of the vortex pair with the aperture and jet, when the influence of vortex shedding into the free streamlines is ignored.

It is clear by symmetry that the direct radiations from each vortex are equal, and we shall therefore continue to express our formulae in terms of the left-hand vortex at the retarded position $\mathbf{x}_o(\tau) \equiv \mathbf{x}_o^-(\tau)$. Thus, using the compact Green's function (37), we find

$$\frac{p_\Gamma(\mathbf{x}, t)}{\left(\frac{\rho_o U \Gamma}{h\sqrt{M}} \right) \left(\frac{h}{|\mathbf{x}|} \right)^{\frac{1}{2}}} \approx -\frac{2}{\pi} \left(\frac{2h}{\pi U} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{v_o(\mathbf{x}_o(\tau)) \chi([t] - \tau)}{\sqrt{[t] - \tau}} \frac{\partial \Phi}{\partial y_\perp}(\mathbf{x}_o(\tau)) \, \mathrm{d}\tau$$

$$= -\frac{1}{\sigma} \left(\frac{2U}{\pi h} \right)^{\frac{1}{2}} \int_{-\infty}^{[t]} \frac{\Im\mathfrak{m}\left\{ \zeta_o/(\zeta_o^2 - 1)^{\frac{1}{2}} \right\}}{\left| \zeta_o + (\zeta_o^2 - 1)^{\frac{1}{2}} \right|^2} \frac{\chi([t] - \tau) \, \mathrm{d}\tau}{\sqrt{[t] - \tau}}, \qquad |\mathbf{x}| \to \infty, \tag{38}$$

where the $\zeta_o = \zeta_o(\tau)$ is the position of the left vortex in the ζ-plane at time τ and $[t] = t - |\mathbf{x}|/c_o$ is the retarded time. The integral expressions here are dimensionless, and can be cast as functions of $U[t]/h$. However, reference to the definition of $\chi(t)$ in (37) will reveal that they are also weakly dependent on the mean-flow Mach number (the dependance being on $\log M$, a consequence of the two-dimensional geometry of the radiation problem). But the main Mach number dependance is signified by $\sqrt{M}$ in the denominator on the left-hand side. Thus, in more general terms, according to Equation (38) $p_\Gamma \sim \rho_o v^2/M^{-\frac{1}{2}}$, which is characteristic of a two-dimensional, hydroacoustic monopole source [5], [6, p. 37]. The monopole evidently arises from the unsteady volume flux through the aperture produced by the approaching vortex pair – although, of course, we have still to take account of the contribution to this flux from the unsteady free-streamline vorticity.

The curves labelled p_Γ in Figure 6 illustrate the nondimensional acoustic waveform predicted by Equation (38) for the two cases $x_\Gamma/h = 0{\cdot}6$ and $0{\cdot}8$. We have taken the jet Mach number $M = 0{\cdot}01$ ($U \sim 15\,\mathrm{m/s}$ in water), but the curves do not change significantly with M. In both cases p_Γ consists of a sharp-fronted pulse that develops rapidly at the retarded time at which the vortex pair passes through the aperture, the maximum amplitude being larger for larger values of x_Γ/h, when the vortices pass close to the aperture edges. The pulse decays slowly at later times (a characteristic of two dimensional radiation problems) and is still significant at retarded times when the vortex pair has travelled many aperture widths into the jet. It may also be noted that $p_\Gamma \equiv 0$ when $x_\Gamma = 0$.

4.3. The overall acoustic radiation

The final integral of (5) supplies the component $p_S(\mathbf{x}, t)$ of the acoustic pressure associated with the bound vorticity on the free streamlines. The contributions from each of the free streamlines are evidently equal, and we shall consider the details for the streamline BC. To evaluate the integral we use the representation (25) of $B(s, \tau)$ on BC. First, the definition (37) gives

$$\frac{\partial \Phi}{\partial y_n} = \frac{\pi}{2\sigma h} \frac{1}{\sqrt{e^{\pi s/\sigma h} - 1}} \quad \text{for} \quad 0 < s < \infty \quad \text{on BC.} \tag{39}$$

The integration with respect to s on BC is performed using the formula ([16, Section 3.912])

$$\int_0^\infty \frac{e^{i\omega s/U}\,ds}{\sqrt{e^{\pi s/\sigma h} - 1}} = \frac{\sigma h}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{2} - \frac{i\sigma h\omega}{\pi U}\right)}{\Gamma\left(1 - \frac{i\sigma h\omega}{\pi U}\right)} \equiv \frac{\sigma h}{F_+(\nu)}. \tag{40}$$

Hence using this and (25) the contribution from BC to the final integral on the right of (5) becomes

$$\int_{\mathrm{BC}} B(s, \tau') \frac{\partial G}{\partial y_n}(\mathbf{x}, \mathbf{y}, t - \tau')\,ds\,d\tau'$$
$$= \frac{\pi\Gamma}{(2\sigma h)^2} \left(\frac{2c_o}{\pi|\mathbf{x}|}\right)^{\frac{1}{2}} \int_{-\infty}^{[t]} \frac{\chi([t] - \tau')}{\sqrt{[t] - \tau'}} \left\{ \int_{-\infty}^{\tau'} |w_o'|^2\, \Im\mathrm{m}\left(\frac{\zeta_o}{(\zeta_o^2 - 1)^{\frac{3}{2}}}\right) d\tau \right\} d\tau', \tag{41}$$

where in the inner integral w_o' and ζ_o are evaluated at the position of the left vortex at time τ (as in the integral of (25)).

Now the equation of motion (8) of the left-hand vortex implies that

$$\frac{d\zeta_o}{d\tau} = \frac{-\pi}{2\sigma h U} |w_o'|^2 \zeta_o. \tag{42}$$

This relation permits the inner integration of (41) to be performed explicitly, so that the right hand side of (41) reduces to

$$\frac{\Gamma U}{2\sigma h}\left(\frac{2c_o}{\pi|\mathbf{x}|}\right)^{\frac{1}{2}}\int_{-\infty}^{[t]}\mathfrak{Im}\left\{\frac{\zeta_o}{(\zeta_o^2-1)^{\frac{1}{2}}}\right\}\frac{\chi([t]-\tau)\,\mathrm{d}\tau}{\sqrt{[t]-\tau}}. \tag{43}$$

When this result is doubled to take account of the equal contribution from the free streamline B′C′, we find that the net acoustic pressure produced by the bound vorticity is given by

$$\frac{p_S(\mathbf{x},t)}{\left(\frac{\rho_o U\Gamma}{h\sqrt{M}}\right)\left(\frac{h}{|\mathbf{x}|}\right)^{\frac{1}{2}}}=\frac{1}{\sigma}\left(\frac{2U}{\pi h}\right)^{\frac{1}{2}}\int_{-\infty}^{[t]}\mathfrak{Im}\left\{\frac{\zeta_o}{(\zeta_o^2-1)^{\frac{1}{2}}}\right\}\frac{\chi([t]-\tau)\,\mathrm{d}\tau}{\sqrt{[t]-\tau}},\qquad |\mathbf{x}|\to\infty. \tag{44}$$

The corresponding acoustic pressure profiles for $x_\Gamma/h=0{\cdot}6,\ 0{\cdot}8$ are plotted in Figure 6, together with the overall acoustic pressure $p=p_\Gamma+p_S$, given (in the same notation) by

$$\frac{p(\mathbf{x},t)}{\left(\frac{\rho_o U\Gamma}{h\sqrt{M}}\right)\left(\frac{h}{|\mathbf{x}|}\right)^{\frac{1}{2}}}=-\frac{1}{\sigma}\left(\frac{2U}{\pi h}\right)^{\frac{1}{2}}\int_{-\infty}^{[t]}\mathfrak{Im}\left\{\frac{\zeta_o}{(\zeta_o^2-1)^{\frac{1}{2}}}\right\}\left[\frac{1}{\left|\zeta_o+(\zeta_o^2-1)^{\frac{1}{2}}\right|^2}-1\right]\frac{\chi([t]-\tau)\,\mathrm{d}\tau}{\sqrt{[t]-\tau}},$$

$$|\mathbf{x}|\to\infty. \tag{45}$$

The profile of p_S is similar in shape but opposite in sign to the directly radiated pressure p_Γ. However, p_S, being produced by vorticity on the free streamlines, can be seen to be significant even at negative retarded times prior to the arrival of the vortex pair at the aperture. It has a characteristic broad pulse shape, with a slowly decaying tail. The tails of p_S and p_Γ ultimately become equal and opposite when $U[t]/h$ becomes large, because

$$\frac{1}{\left|\zeta_o+(\zeta_o^2-1)^{\frac{1}{2}}\right|^2}\to 1\quad\text{as}\quad\frac{U[t]}{h}\to+\infty.$$

But it is apparent from Figure 6 that the radiation is strongly dominated by the free-streamline contribution prior to the onset of this asymptotic limit. Figure 7 indicates that the peak overall radiation occurs for $x_\Gamma/h\sim 0{\cdot}6$, which is consistent with Figure 4, where it was seen that the maximum amount of bound vorticity occurs for such values of x_Γ/h. The separate peak contributions of p_Γ and p_S are larger when x_Γ/h increases towards 1 (so that vortices pass close to 'singularities' at the aperture edges), but the overall net acoustic pressure still does not exceed that for $x_\Gamma/h\sim 0{\cdot}6$.

In the formal limit in which $x_\Gamma/h\sim 1$ each vortex follows a trajectory immediately adjacent to the wetted surface of one of the walls and subsequently convects along a free streamline. The overall acoustic pressure given by (45) then vanishes identically for all retarded times τ. (The first factor in the integrand vanishes for $\tau<0$ and the term in square braces vanishes for $\tau>0$.) This prediction is a consequence of linear theory, that requires the vortices $\pm\Gamma$ and the shed vorticity to convect at the same speed U on the free streamlines. It is analogous to the corresponding treatment of trailing edge noise generated by vorticity convected in uniform flow over a knife edge [6, Section 3.5], [17], where linear theory implies that both impinging and shed vorticity convect at the same velocity, and the acoustic pressures produced by each are always equal and opposite.

5. Conclusion

Low-Mach-number aeroacoustic sources of monopole type usually occur in regions where turbulent flow can create a net volume flux into or out of the acoustic medium, such as in the vicinity of an aperture in a boundary. The amplitude of the acoustic pressure radiated (in three dimensions) is then proportional to $\rho_o v^2$, v being a typical velocity in the source flow, whereas the more usual sources of quadrupole or dipole type, associated with free-field turbulence and turbulence interacting with acoustically compact boundary elements, produce characteristically smaller pressures scaling, respectively, like $\rho_o v^2 M^2$ and $\rho_o v^2 M$.

The acoustic amplitude is typically increased by a factor $1/\sqrt{M}$ in cases where the source region is two-dimensional over distances comparable to the acoustic wavelength, which frequently allows three-dimensional scaling laws to be inferred from the solutions of two-dimensional problems. This is the case for the problem examined in this paper of sound production during high-Reynolds-number convection of a vortex pair in an aqueous jet exhausting from a large vessel through a slit-like wall aperture. Our detailed analysis has revealed how the monopole sound produced arises from a competition between dynamically opposed vortex sources: the acoustic pulse generated by the incident vorticity is countered by the production of sound of opposite phase by vorticity shed from the edges of the jet outlet and swept out of the vessel in the jet shear layer. It appears from our calculations that the shear-layer vorticity is actually the dominant source of the sound, radiating pressure fluctuations back into the vessel whose amplitude scales like $\rho_o v^2/\sqrt{M}$, and therefore like $\rho_o v^2$ in the corresponding flow from a three-dimensional aperture.

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